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REDUCTION OF A CLASS OF INVERSE HEAT-CONDUCTION PROBLEMS TO DIRECT INITIAL/BOUNDARY-VALUE PROBLEMS

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The authors present a method of reducing inverse problems of recovery of boundary heat fluxes by means of data of integral or differential temperature measurements on the boundary to direct initial/boundary-value problems.

In the present work consideration is given to two problems of identification of unsteady heat fluxes on part of the boundary of a body by using data of integral or differential temperature measurements on the boundary. It is assumed that the heat transfer is described by the two-dimensional unsteady heat-conduction equation. The indicated problems belong to inverse heat-conduction problems. We show that they are reduced to direct initial/boundary-value problems for the heat-conduction equation with nonclassical (integrodifferential) boundary conditions in the case of integral temperature measurements and classical boundary conditions in the case of differential measurements. The suggested approach of reduction of the inverse problems to direct ones is based on the works [1-3].

1. Heat-Flux Recovery by Means of Data of Integral Temperature Measurement. In the two-dimensional region $\Omega = \{(x, y)\colon 0 \le x \le a,\ 0 \le y \le b\}$ we will consider the parabolic equation

$$c(x, y)\, u_t = \operatorname{div}(\lambda(x, y) \operatorname{grad} u)\,, \quad u = u(x, y, t)\,, \quad t \ge 0\,, \tag{1}$$

with the initial and boundary conditions

$$u(x, y, 0) = u_0(x, y)\,, \quad u_y(x, 0, t) = u_x(0, y, t) = u_x(a, y, t) = 0\,, \tag{2}$$

$$-\lambda(x, b)\, u_y(x, b, t) = q(x, t)\,, \tag{3}$$

where c, λ, q are prescribed sufficiently smooth functions ($c > 0$, $\lambda > 0$).

The direct problem is formulated in the form (1)-(3). Consideration will be given to the inverse problem in which in addition to $u(x, y, t)$ the right-hand side $q(x, t)$ of boundary condition (3) is unknown. We will assume that $q(x, t)$ can be represented as

$$q(x, t) = \eta(t)\, \psi(x)\,, \tag{4}$$

where the function $\psi(x)$ is prescribed, while the time-dependent boundary source, i.e., the function $\eta(t)$ in representation (4), is unknown. This dependence is recovered using additional information, i.e., the function $\varphi(t)$:

$$\varphi(t) = \int_0^a p(s)\, u(s, b, t)\, ds\,. \tag{5}$$

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In heat-transfer problems the function $\eta(t)$ has the meaning of the time-dependent amplitude of the boundary heat flux $q(x, t)$, while $\varphi(t)$ is the averaged temperature with the weight function $p(x)$ on the surface $y = b$. Here, the inverse problem (1)-(5) can be interpreted either as the problem of identification of the heat flux $q(x, t)$ or as the problem of programmed control of the averaged temperature (5), for which $\varphi(t)$ is the specified program.

Now we will give some examples of the weight function $p(x)$.

1) If

$$p(x) = \begin{cases} \dfrac{1}{x_1 - x_0}, & x \in [x_0, x_1] \subseteq [0, a], \\ 0, & x \notin [x_0, x_1], \end{cases}$$

then $\varphi(t)$ is the average temperature at the time t in the region $y = b$, $x \in [x_0, x_1]$.

2) If $p(x) = \delta(x - x_0)$ $(0 < x_0 < a)$ is the Dirac function, then $\varphi(t)$ is the temperature at the point (x_0, b).

3) Let $\varepsilon < \max\{x_0, a - x_0\}$. We set

$$p(x) = p_\varepsilon(x) = \begin{cases} \varepsilon^{-2}(\varepsilon - x_0 + x), & x \in [x_0 - \varepsilon, x_0] \\ \varepsilon^{-2}(\varepsilon + x_0 - x), & x \in]x_0, x_0 + \varepsilon] \\ 0, & x \notin]x_0 - \varepsilon, x_0 + \varepsilon[\end{cases}$$

The sequence of functions $p_\varepsilon(x)$ $(\varepsilon \to 0)$ can be considered as a regularization of the generalized function $\delta(x - x_0)$.

We transform the inverse problem (1)-(5) into a direct one for Eq. (1) with integrodifferential boundary conditions.

Let

$$b(x) := \frac{\psi(x)}{\lambda(x, b)}, \quad k := \int_0^a b^2(s)\, ds .$$

Equation (3), where $q(x, t)$ has the form of (4), is equivalent to the system of equations

$$k u_y(x, b, t) - b(x) \int_0^a b(s)\, u_y(s, b, t)\, ds = 0, \tag{6}$$

$$k\eta(t) + \int_0^a b(s)\, u_y(s, b, t)\, ds = 0. \tag{7}$$

Indeed, (3) obviously follows from (6), (7). To prove the reverse statement, we will integrate the equality

$$b(x)\, u_y(x, b, t) + b^2(x)\, \eta(t) = 0$$

equivalent to (3) with respect to x within the limits $0 \le x \le a$. As a result, we arrive at equality (7). Now we will express the function in (7) as

$$\eta(t) = \frac{-1}{k} \int_0^a b(s)\, u_y(s, b, t)\, ds . \tag{8}$$

After substitution of (8) into (3), we obtain (6).

Thus, the sought function $\eta(t)$ is determined, according to (8), in terms of the function $u(x, y, t)$, which, in turn, is the solution of parabolic equation (1) with integrodifferential initial/boundary conditions (2), (5), and (6).

Next, we note that the following equality follows from conditions (5) and (6):

$$ku_y(x, b, t) - b(x)\left(\int_0^a b(s)\, u_y(s, b, t)\, ds + \int_0^a p(s)\, u(s, b, t)\, ds\right) = b(x)\, \varphi(t)\,. \tag{9}$$

Multiplying (9) by $b(x)$ and integrating the obtained relation with respect to x within the limits $0 \le x \le a$, we obtain equality (5). Moreover, from (5) and (9), relation (6) follows. We have shown that the system of boundary conditions (5) and (6) is equivalent to the single boundary condition (9).

Finally, we can state that the pair $(\eta(t), u(x, y, t))$, i.e., the solution of the inverse problem (1)-(5), is determined by formula (8) and the direct initial/boundary-value problem (1), (2), and (9).

We note that if we set $p(x) = \delta(x - x_0)$, then the boundary condition (9) acquires the form

$$ku_y(x, b, t) - b(x)\left(\int_0^a b(s)\, u_y(s, b, t)\, ds + u(x_0, b, t)\right) = b(x)\, \varphi(t)\,.$$

The system of equations (1), (2), and (9) is an initial/boundary-value problem with nonclassical boundary conditions. Such problems appear in different fields of science and technology, and beginning with the well-known studies by V. A. Steklov and A. N. Tikhonov, they have attracted the interest of many authors (see, e.g., [4-7]). In particular, from the works [6, 7] it follows that if the function $p(x)$ has the form

$$p(x) = p_0(x) + \sum_{i=1}^{k} p_i\, \delta(x - x_i)\,, \quad x_i \in \,]0, a[\,, \tag{10}$$

where $p_0(x)$ is a piecewise-continuous function, $p_0(x) \ge 0$, $b(x) \ge 0\ \forall x \in [0, a]$, $p_i \ge 0\ \ \forall i \in \{1, ..., k\}$, then the solution $u(x, y, t)$ of system (1), (2), (9) depends continuously (relative to the metric of the space of continuous functions) on the initial and boundary data. The existence and uniqueness of the solution of the inverse problem (1)-(5) and (10) follow from this.

2. Heat-Flux Recovery by Means of Data of Differential Temperature Measurement. We consider the initial/boundary-value problem

$$u_t = u_{xx} + u_{zz}\,, \quad u(x, z, 0) = u_0(x, z)\,, \quad x \in (-\infty, \infty)\,, \quad z \in [0, 1]\,, \tag{11}$$

$$u_z(x, 0, t) = 0\,, \tag{12}$$

$$-u_z(x, 1, t) = q(x, t)\,. \tag{13}$$

The inverse problem involves determination of the heat flux $q(x, t)$ on the surface $z = 1$ from data of temperature-difference measurements

$$u(x, 1, t) - u(x, 0, t) = y(x, t) \tag{14}$$

(the differential-thermocouple method [8]).

From (11)-(14) it directly follows that the solution $(q(x, t), u(x, z, t))$ of the inverse problem is determined by formula (13) and the direct initial/boundary-value problem (11), (12), and (14) with nonclassical boundary condition (14). The spectral boundary-value problem

$$\mu\tilde{u} = \tilde{u}_{xx} + \tilde{u}_{zz}\,, \quad \tilde{u}(x, 0) - \tilde{u}(x, 1) = 0\,, \quad u_z(x, 0) = 0\,,$$

that corresponds to (11), (12), and (14) is not self-conjugate. This fact explains the difficulties encountered in analytical and numerical solution of the system (11), (12), and (14).

In the one-dimensional case ($u = u(z, t)$), the initial/boundary-value problem (11), (12), and (14) was studied in [1, 9]. In [9] it was shown that in this case the Green function of the boundary-value problem (11), (12), and (14) can be constructed by the method of separation of variables. In [1], an alternative approach was suggested that reduces construction of the Green function to successive solution of classical initial/boundary-value problems. Following this approach, we can represent the solution $u(x, z, t)$ of Eq. (11) in the form

$$u(x, z, t) = u^+(x, z, t) + u^-(x, z, t)\,, \tag{15}$$

$$u^+(x, z, t) = \frac{u(x, z, t) + u(x, 1 - z, t)}{2}\,,$$

$$u^-(x, z, t) = \frac{u(x, z, t) - u(x, 1 - z, t)}{2}\,.$$

Substituting (15) into (11)-(14) and considering that $u \equiv 0 \Leftrightarrow \{u^+ \equiv 0,\ u^- \equiv 0\}$, we obtain the system of equations $u_t^+ = u_{xx}^+ + u_{zz}^+$, $u^+(x, z, 0) = u_0^+(x, z)$, $-2u_z^+(x, 1, t) = q(x, t)$ for the function $u^+(x, z, t)$ and the system of equations

$$u_t^- = u_{xx}^- + u_{zz}^-\,, \quad u^-(x, z, 0) = u_0^-(x, z)\,, \tag{16}$$

$$2u^-(x, 1, t) = y(x, t) \tag{17}$$

for the function $u^-(x, z, t)$. Here, it is obvious that the following equality is fulfilled:

$$q(x, t) = -\,2u_x^-(x, 1, t)\,. \tag{18}$$

The system (16) and (17) is equivalent to the classical initial/boundary-value problem

$$v_t = v_{xx} + v_{zz}\,, \quad v(x, z, 0) = u_0^-(x, z)\,, \tag{19}$$

$$2v(x, 0, t) = -\,y(x, t)\,, \quad 2v(x, 1, t) = y(x, t)\,. \tag{20}$$

Since here $v(x, z, t) = u^-(x, z, t)$, from (18) it follows that

$$q(x, t) = -\,2v_z(x, 1, t)\,. \tag{21}$$

Thus, the solution $q(x, t)$ of the inverse problem is determined by formula (21), where $v(x, z, t)$ is the solution of the classical initial/boundary-value problem (19) and (20). Let $G(x, z, \xi, \theta, t)$ be the Green function of the problem (19) and (20). Then the solution of the inverse problem is

$$q(x, t) = -\frac{\partial}{\partial z}\left(2\int_{-\infty}^{\infty}\int_0^1 G(x, z, \xi, \theta, t)\, u_0^-(\xi, \theta)\, d\xi\, d\theta - \right.$$

$$\left. -\int_0^t\int_{-\infty}^{\infty} \left(G_\theta(x, z, \xi, 0, t - s) - G_\theta(x, z, \xi, 1, t - s)\right) y(\xi, s)\, d\xi\, ds\right).$$

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